PAPER - I : MODEL PAPER - 06

(BASED ON MARCH 2018**)**

MATHEMATICS & STATISTICS

COMMERCE

TIME : 1 HR 30 MIN **MARKS : 40**

Q1. (A) Attempt any six of the following **(12)**

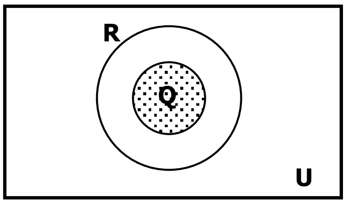
01. Draw Venn Diagram for the truth of the following statements

a) All rational numbers are real numbers

Q ≡ set of all rational numbers

R ≡ set of all real numbers

U ≡ set of all numbers (complex)

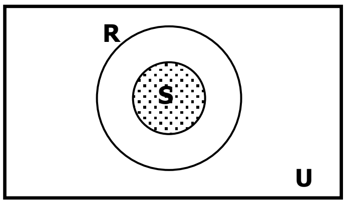


b) Some rectangles are squares

R ≡ set of all rectangles

S ≡ set of all squares

U ≡ set of all quadrilaterals



02. Find the inverse of the matrix $A = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$ using elementary row transformation

SOLUTION

$|A| = 15 - 8 = 7 \neq 0$
Hence A^{-1} exist

$AA^{-1} = I$

$$\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$R_1 - R_2$

$$\begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$R_2 - 2R_1$

$$\begin{pmatrix} 1 & -1 \\ 0 & 7 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$R_2/7$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & -1 \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} A^{-1} = \frac{1}{7}\begin{pmatrix} 7 & -7 \\ -2 & 3 \end{pmatrix}$$

$R_1 + R_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} = \frac{1}{7}\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$I \,.\, A^{-1} = \frac{1}{7}\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

03. $f(x) = \dfrac{5^x - 3^x}{2^x - 1}$; $x \neq 0$

$= \log(5/3) / \log 2$; $x = 0$

Discuss continuity at $x = 0$

STEP 1

$$\lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{5^x - 3^x}{2^x - 1}$$

$$= \lim_{x \to 0} \frac{5^x - 1 - 3^x + 1}{2^x - 1}$$

$$= \lim_{x \to 0} \frac{(5^x - 1) - (3^x - 1)}{2^x - 1}$$

Divide Numerator & Denominator by x , $x \to 0$, $x \neq 0$

$$= \lim_{x \to 0} \frac{\dfrac{(5^x - 1) - (3^x - 1)}{x}}{\dfrac{2^x - 1}{x}}$$

$$= \lim_{x \to 0} \frac{\dfrac{5^x - 1}{x} - \dfrac{3^x - 1}{x}}{\dfrac{2^x - 1}{x}}$$

$$= \frac{\log 5 - \log 3}{\log 2}$$

$$= \frac{\log(5/3)}{\log 2}$$

STEP 2 :

$f(0) = \log(5/3) / \log 2$……… given

STEP 3 :

$$f(0) = \lim_{x \to 0} f(x)$$

$\therefore$ f is continuous at $x = 0$

04. Find dy/dx if $y = \cos^{-1} \sqrt{x}$

SOLUTION

$y = \cos^{-1} \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x}} \frac{d}{dx}\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x(1 - x)}}$$

05. find values of x for which

$f(x) = x + 9/x$ is decreasing

SOLUTION

For $f(x)$ decreasing ,

$f'(x) < 0$

$$1 - \frac{9}{x^2} < 0$$

$$\frac{x^2 - 9}{x^2} < 0$$

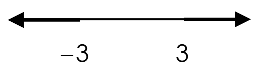
$x^2 - 9 < 0$

$(x - 3).(x + 3) < 0$

CASE 1 :

$x - 3 > 0$ & $x + 3 < 0$

$x > 3$ & $x < -3$

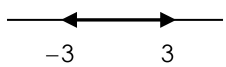


NOT POSSIBLE SO DISCARD

CASE 2 :

$x - 3 < 0$ & $x + 3 > 0$

$x < 3$ & $x > -3$



$-3 < x < 3$

f **is decreasing for** $\boldsymbol{x \in (-3, 3), x \neq 0}$

06.

$$\int \frac{x + 1}{x.(x + \log x)}\, dx$$

SOLUTION

PUT

$x + \log x = t$

$1 + \frac{1}{x} . dx = dt$

$\frac{x + 1}{x} . dx = dt$

NOW THE SUM IS

$= \int \frac{1}{t}\, dt$

$= \log | t | + c$

$= \log | x + \log x | + c$

07. $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$

SOLUTION

COFACTORS

$A_{11} = (-1)^{1+1}(4) = 4$

$A_{12} = (-1)^{1+2}(-3) = 3$

$A_{21} = (-1)^{2+1}(2) = -2$

$A_{22} = (-1)^{2+2}(-1) = -1$

08. $\int \frac{1}{\sqrt{2x - x^2}}\, dx$

$= \int \frac{1}{\sqrt{0 - (x^2 - 2x)}}\, dx$

$= \int \frac{1}{\sqrt{0 - (x^2 - 2x + 1) + 1}}\, dx$

$= \int \frac{1}{\sqrt{1 - (x - 1)^2}}\, dx$

$= \int \frac{1}{\sqrt{1^2 - (x - 1)^2}}\, dt$

$= \sin^{-1}(x - 1) + c$

Q2.

(A) Attempt any TWO of the following (06)

01. $f(x) = \frac{x^2}{5^x + 5^{-x} - 2}$; $x \neq 0$.

Find f(0) if f(x) is CONTINOUS at x = 0

SOLUTION

STEP 1

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \frac{x^2}{5^x + 5^{-x} - 2}$

$= \lim_{x \to 0} \frac{x^2}{5^x + \frac{1}{5^x} - 2}$

$= \lim_{x \to 0} \frac{x^2}{\frac{(5^x)^2 + 1 - 2.5^x}{5^x}}$

$= \lim_{x \to 0} \frac{x^2}{\frac{(5^x - 1)^2}{5^x}}$

$= \lim_{x \to 0} \frac{5^x}{\frac{(5^x - 1)^2}{x^2}}$

$= \lim_{x \to 0} \frac{5^x}{\left(\frac{5^x - 1}{x}\right)^2}$

$= \frac{5^0}{(\log 5)^2}$

$= \frac{1}{(\log 5)^2}$

STEP 2

Since *f* is continuous at x = 0

$$f(0) = \lim_{x \to 0} f(x) = \frac{1}{(\log 5)^2}$$

02. Write the converse, inverse and the contra-positive of the statement

"if two triangles are not congruent then their areas are not equal"

$P \to Q$

if two triangles are not congruent then their areas are not equal

CONVERSE

$Q \to P$

if areas are not equal then two triangles are not congruent

CONTRAPOSITIVE

$\sim Q \to \sim P$

if areas are equal then two triangles are congruent

INVERSE

$\sim P \to \sim Q$

if two triangles are congruent then their areas are equal

03. if $x = \cos^2\theta$ and $y = \cot\theta$ then find dy/dx at $\theta = \pi/4$

STEP 1

$x = \cos^2\theta$

$$\therefore \frac{dx}{d\theta} = 2\cos\theta \frac{d}{d\theta}\cos\theta$$

$$= -2\cos\theta \sin\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta = \pi/4} = -2\cos \pi/4 \sin \pi/4$$

$$= -2 \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = -1$$

STEP 2

$y = \cot\theta$

$$\frac{dy}{d\theta} = -\operatorname{cosec}^2\theta$$

$$\left.\frac{dy}{d\theta}\right|_{\theta = \pi/4} = -\operatorname{cosec}^2 \pi/4 = -2$$

STEP 3

$$\left.\frac{dy}{dx}\right|_{\theta = \pi/4} \quad \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta = \pi/4} = 2$$

(B) Attempt any TWO of the following (08)

01. The sum of three numbers is 6. If we multiply the third number by 3 and add it to the second number we get 11. By adding first and third numbers we get a number which is double than the second number. Use these information & find a system of linear equations. Find these three numbers using MATRICES

SOLUTION

Let three numbers be x, y & z

As per the given condition

$$x + y + z = 6 \quad \text{.... (1)}$$

$$y + 3z = 11 \quad \text{.... (2)}$$

$$x + z = 2y$$

$$x - 2y + z = 0 \quad \text{.... (3)}$$

$AX = B$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ 0 \end{pmatrix}$$

$R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x + y + z \\ y + 3z \\ -3y \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -6 \end{pmatrix}$$

BY EQUALITY OF TWO MATRICES

$-3y = -3 \qquad \therefore y = 2$

$y + 3z = 11$

subs $y = 2 \qquad \therefore z = 3$

$x + y + z = 6$

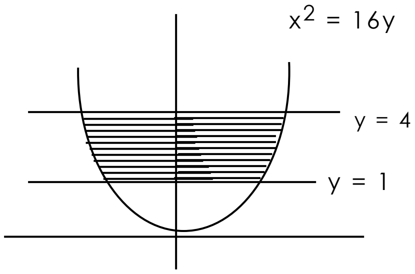
subs $y = 2$ & $z = 3 \quad \therefore x = 1$

Hence the three numbers are 1, 2 & 3

02.

Find the area of the region bounded by the curve $x^2 = 16y$ and the lines $y = 1$; $y = 4$

SOLUTION



$$A = 2\int_1^4 x \; dy$$

$$= 2\int_1^4 \sqrt{16y} \; dy$$

$$= 2\int_1^4 4\sqrt{y} \; dy$$

$$= 8 \int_1^4 y^{1/2} \, dy$$

$$= 8 \left[\frac{y^{3/2}}{\frac{3}{2}} \right]_1^4$$

$$= \frac{16}{3} \left(y^{3/2} \right)_1^4$$

$$= \frac{16}{3} \left(4^{3/2} - 1^{3/2} \right)$$

$$= \frac{16}{3} \left(2^{2.3/2} - 1 \right)$$

$$= \frac{16}{3} \left(2^3 - 1 \right)$$

$$= \frac{16}{3} (8 - 1)$$

$$= \frac{112}{3} \text{ sq. units}$$

03.

Divide 16 into two parts such that sum of their squares is minimum

SOLUTION

Let one part be x and other part be $16 - x$

STEP 1

$$f(x) = x^2 + (16 - x)^2$$
$$= x^2 + 256 - 32x + x^2$$
$$= 2x^2 - 32x + 256$$

STEP 2

$$f'(x) = 4x - 32$$
$$f''(x) = 4$$

STEP 3

$$f'(x) = 0$$
$$4x - 32 = 0 \quad \therefore \; x = 8$$

STEP 4

$$f''(x)\; x = 8 = 4 > 0$$

$f(x)$ is minimum at $x = 8$

Hence divide 16 into 8 , 8

Q3. (A) Attempt any TWO of the following

01. $f(x) = \dfrac{x^3 - 64}{\sqrt{x^2+9} - 5} \quad ; \quad x \neq 4$

$\quad\quad\quad = 10 \quad ; \quad x = 4$

Discuss continuity at $x = 4$

SOLUTION

STEP 1

$$\lim_{x \to 4} f(x)$$

$$= \lim_{x \to 4} \frac{x^3 - 64}{\sqrt{x^2+9} - 5}$$

$$= \lim_{x \to 4} \frac{x^3 - 64}{\sqrt{x^2+9} - 5} \cdot \frac{\sqrt{x^2+9} + 5}{\sqrt{x^2+9} + 5}$$

$$= \lim_{x \to 4} \frac{x^3 - 4^3}{x^2 + 9 - 25} \cdot \frac{\sqrt{x^2+9} + 5}{1}$$

$$= \lim_{x \to 4} \frac{(x^3 - 4^3)}{x^2 - 16} \cdot \frac{\sqrt{x^2+9} + 5}{1}$$

$$= \lim_{x \to 4} \frac{\cancel{(x - 4)}(x^2+4x+16)}{\cancel{(x - 4)}(x + 4)} \cdot \frac{\sqrt{x^2+9} + 5}{1}$$

$x - 4 \neq 0$

$$= \lim_{x \to 4} \frac{(x^2+4x+16)}{(x + 4)} \cdot \frac{\sqrt{x^2+9} + 5}{1}$$

$$= \frac{(4^2+4\cdot4+16)}{(4 + 4)} \cdot \frac{\sqrt{4^2+9} + 5}{1}$$

$$= \frac{(16+16+16)}{8} \cdot \frac{\sqrt{16+9} + 5}{1}$$

$$= \frac{(48)}{8} \cdot \frac{5 + 5}{1}$$

$$= (6)\ (10)$$

$$= 60$$

STEP 2

$f(4) = 10$ ……… given

STEP 3

$f(4) \neq \lim\limits_{x \to 4} f(x)$; f is discontinuous at $x = 4$

STEP 4

Removable Discontinuity

f can be made continuous at $x = 4$ by redefining it as

$f(x) = \dfrac{x^3 - 64}{\sqrt{x^2+9} - 5} \quad ; \quad x \neq 4$

$\quad\quad\quad = 240 \quad ; \quad x = 4$

02. $x = \dfrac{2t}{1 + t^2} \quad ; \quad y = \dfrac{1 - t^2}{1 + t^2}$

Show that $\dfrac{dy}{dx} = \dfrac{-x}{y}$

SOLUTION

✓ $x = \dfrac{2t}{1 + t^2}$ Put $t = \tan\theta$

$$x = \frac{2\tan\theta}{1 + \tan^2\theta}$$

$$x = \sin 2\theta \quad \text{…………. (1)}$$

$$\frac{dx}{d\theta} = \cos 2\theta \ \frac{d}{d\theta} 2\theta$$

$$= 2\cos2\theta$$

✓ $y = \dfrac{1 - t^2}{1 + t^2}$ Put $t = \tan\theta$

$$y = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$y = \cos 2\theta \quad \text{………….. (2)}$$

$$\frac{dy}{d\theta} = -\sin 2\theta \ \frac{d}{d\theta} 2\theta$$

$$= -2\sin2\theta$$

✓ $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

$$= \frac{-2\sin2\theta}{2\cos2\theta}$$

$$= -\frac{\sin 2\theta}{\cos 2\theta}$$

$$= -\frac{x}{y} \quad \text{….. From 1 \& 2}$$

03. Using truth table show that

$\sim (p \rightarrow \sim q) \equiv p \wedge q$

SOLUTION

p	q	~q	$p \rightarrow \sim q$	$\sim (p \rightarrow \sim q)$ COLA	$p \wedge q$ COLB
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

Since truth values in col A and col B are identical , $\sim (p \rightarrow \sim q) \equiv p \wedge q$

(B) Attempt any TWO of the following (08)

01. Evaluate : $\displaystyle\int \frac{\sin x}{\sqrt{\cos^2 x - 2\cos x - 3}}\, dx$

SOLUTION

PUT $\cos x = t$

$-\sin x.\, dx = dt$

$\sin x.\, dx = -dt$

THE SUM IS

$$= \int \frac{-1}{\sqrt{t^2 - 2t - 3}}\, dt$$

$$= \int \frac{-1}{\sqrt{t^2 - 2t + 1 - 3 - 1}}\, dt$$

$$= \int \frac{-1}{\sqrt{(t-1)^2 - 4}}\, dt$$

$$= \int \frac{-1}{\sqrt{(t-1)^2 - 2^2}}\, dt$$

$$= \log \left| t + \sqrt{t^2 - a^2} \right| + c$$

$$= -\log \left| t - 1 + \sqrt{(t-1)^2 - 2^2} \right| + c$$

$$= -\log \left| t - 1 + \sqrt{t^2 - 2t - 3} \right| + c$$

$$= -\log \left| \cos x - 1 + \sqrt{\cos^2 x - 2\cos x - 3} \right| + c$$

02.

cost of x benches is given as $C = 200 + {}^{x}/_{5}$ and are then sold @ $15 - {}^{x}/_{2000}$ each . Find the number of benches to be sold to make profit maximum

SOLUTION

STEP 1

$C = 200 + \frac{x}{5}$, $R = px = 15x - \frac{x^2}{2000}$

$$\pi = R - C$$

$$= 15x - \frac{x^2}{2000} - 200 - \frac{x}{5}$$

$$= 15x - \frac{x}{5} - \frac{x^2}{2000} - 200$$

$$= \frac{74x}{5} - \frac{x^2}{2000} - 200$$

STEP 2

$$\frac{d\pi}{dx} = \frac{74}{5} - \frac{2x}{2000}$$

$$\frac{d^2\pi}{dx^2} = \frac{-2}{2000}$$

STEP 3

$$\frac{d\pi}{dx} = 0$$

$$\frac{74}{5} - \frac{2x}{2000} = 0$$

$$\frac{74}{5} = \frac{2x}{2000} \qquad x = 14800$$

STEP 4

$$\left.\frac{d^2\pi}{dx^2}\right|_{x = 14800} = \frac{-2}{2000} < 0$$

hence profit is maximum at x = 14800

ans : number of benches to be sold to make profit maximum = 14800 units

03. $\displaystyle\int_{2}^{3} \frac{x}{(x+2)(x+3)}\,dx$

$$\frac{x}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$x = A(x+3) + B(x+2)$$

Put $x = -3$

$-3 = B(-3+2)$

$-3 = B(-1)$

$3 = B$

Put $x = -2$

$-2 = A(-2+3)$

$-2 = A(1)$

$-2 = A$

HENCE

$$\frac{x}{(x+2)(x+3)} = \frac{-2}{x+2} + \frac{3}{x+3}$$

BACK IN THE SUM

$$= \int_{2}^{3} \left(\frac{-2}{x+2} + \frac{3}{x+3} \right) dx$$

$$= \Big(-2 \log |x+2| + 3 \log |x+3| \Big)_{2}^{3}$$

$$= \big(-2 \log 5 + 3 \log 6\big) - \big(-2 \log 4 + 3 \log 5\big)$$

$$= -2 \log 5 + 3 \log 6 + 2 \log 4 - 3 \log 5$$

$$= 2 \log 4 + 3 \log 6 - 5 \log 5$$

$$= \log 4^2 + \log 6^3 - \log 4^5$$

$$= \log 16 + \log 216 - \log 3125$$

$$= \log \left(\frac{16 \times 216}{3125} \right)$$

$$= \log \left(\frac{3456}{3125} \right)$$

DO NOT STOP

GET READY FOR NEXT